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On structures of modular adjacency algebras of Johnson schemes

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ABSTRACT

In this paper, we consider algebras over a field of characteristic p, which are generated by adjacency algebras of Johnson schemes. If the algebra is semisimple, the structure is the same as that of the well-known Bose-Mesner algebras. We determine the structure of the algebra when it is not semisimple.

1. Introduction

Let G be an association scheme and K a field. We consider an algebra generated by adjacency matrices of G over K, the so-called adjacency algebra KG of G over K [1,5]. If K has characteristic 0, KG is semisimple. Many researchers have studied this case and there are many results [1,3]. However, there are few for the case where K has positive characteristic. We call such KG modular adjacency algebras. When KG is semisimple, we have the case of characteristic 0. The structure is isomorphic to a direct sum of Ks; hence, we shall focus on the case where KG is not semisimple. Yoshikawa described modular adjacency algebras of Hamming schemes as factor algebras of a polynomial ring by ideals [11]. These algebras are local algebras; hence, there is only one block. Hanaki and Yoshikawa studied association schemes of class 2 [7]. If an algebra is a factor algebra, we know whether or not the number of isomorphism classes of finite dimensional indecomposable modules of the algebra is finite. If it is finite, we say that the algebra is a finite-representation type. Actually, Hanaki and Hieda considered representations of strongly p'-valenced Schurian schemes of order pq using Brauer tree algebras [6]. They showed that the modular adjacency algebra of a Johnson scheme $J(p,t)$ is a finite-representation type. In this paper, we describe structures of modular adjacency algebras of Johnson schemes. These algebras are not always local algebras. We know when a modular adjacency algebras of Johnson scheme is a finite-representation type. This result helps us to construct the modular representation theory of association schemes.

2. Preliminaries

Let M be a finite set with m elements. Consider a collection of subsets of M, $\binom{M}{n} = \{N \subset M \mid |N| = n\}$, for $0 < n \leq m/2$. We define the Johnson distance $\rho(N_1, N_2) = n - |N_1 \cap N_2|$, for two subsets $N_1, N_2 \in \binom{M}{n}$ and relations $R_i = \{(N_1, N_2) \mid \rho(N_1, N_2) = i\}$. It is known that $\mathfrak{X} = \left(\binom{M}{n}, \{R_i\}_{0 \leq i \leq n}\right)$ forms an association scheme [1]. $\mathfrak{X} = J(m,n)$ is called the Johnson scheme. Let $\{D_i\}_{i=0,\dots,n}$ be adjacency matrices with respect to relations. Let v_i denote the valency of D_i [1]. Let us denote the adjacency algebra of $\mathfrak{X}$ over a field K by $K\mathfrak{X}$. We consider the character table P. Let (K, R, F) be a

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splitting p-modular system for the adjacency algebra, and (π) be the maximal ideal of R. We denote the image of the natural epimorphism $R \to F$ by $*$. For details of a p-modular system, the reader is referred to [9]. Since

$$R\mathfrak{X}/\pi R\mathfrak{X} \cong F\mathfrak{X}$$

and $\pi R\mathfrak{X} \subset J(R\mathfrak{X})$, idempotents of $F\mathfrak{X}$ are liftable to idempotents of $R\mathfrak{X}$. In $F\mathfrak{X}$, 1 can be expressed as the sum of central primitive idempotents:

$$1 = f_0 + \cdots + f_s.$$

Accordingly, the central primitive idempotents decomposition of 1 in $R\mathfrak{X}$ is

$$1 = e_0 + \cdots + e_s,$$

where $e_i^* = f_i$, $(0 \le i \le s)$.

Letting b_i denote $e_i R\mathfrak{X}$ and b_i^* denote $e_i^* F\mathfrak{X}$, then we obtain the indecomposable decomposition of $R\mathfrak{X}$ into two-sided ideals:

$$R\mathfrak{X} = b_0 \oplus \cdots \oplus b_s.$$

The corresponding indecomposable decomposition of $F\mathfrak{X}$ into two-sided ideals is

$$F\mathfrak{X} = b_0^* \oplus \cdots \oplus b_s^*.$$

We call b_i (or b_i^*) a block of $R\mathfrak{X}$ (or $F\mathfrak{X}$). Mapping each adjacency matrix D_i to v_i induces an algebra homomorphism from $F\mathfrak{X}$ to F. This representation is usually called the *trivial representation* of $F\mathfrak{X}$. We assume that χ_0 is the trivial representation, and χ_0 belongs to b_0. We call b_0 the *principal block* of $\mathfrak{X}$. We use the following lemma often.

Lemma 1 (*Lucas,[2]*)**.** *Let p be a prime and put $m = a_0 + d_1 p + \cdots + d_k p^k$ and $n = b_0 + b_1 p + \cdots + b_k p^k$, where $0 \le a_i, b_i < p (0 \le i \le k-1)$. Then*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \pmod p.$$

3. Epimorphism

Let C_i be a linear combination of adjacency matrices such that

$$C_i := \sum_{u=i}^{n} \binom{u}{i} D_{n-u}, \quad (i = 0, 1, \ldots, n). \tag{1}$$

Since the coefficient of D_{n-i} is 1,

$$\bigoplus_{i=0}^{n} FD_i = \bigoplus_{i=0}^{n} FC_i. \tag{2}$$

Then $C_0, \ldots, C_n$ form another basis of the adjacency algebra of the Johnson scheme [3],

$$C_r C_s = \sum_{t=0}^{\min(r,s)} \binom{n-t}{r-t} \binom{n-t}{s-t} \binom{m-r-s}{m-n-t} C_t. \tag{3}$$

Theorem 2. *For each positive integer n, $\varphi_n : FJ(2n+k, n) \longrightarrow FJ(2(n-1)+k, n-1) (C_i^n \mapsto C_{i-1}^{n-1})$ is an algebra epimorphism.*

Proof. Let C_i^n (or C_i^{n-1}) be an F-basis of $FJ(2n+k, n)$ (or $FJ(2(n-1)+k, n-1)$) as per the above definition (1), and we put $C_{-1}^{n-1} = 0$. It is clear that the map φ_n is surjective by the correspondence $\varphi_n(C_i^n) = C_{i-1}^{n-1}$.

$$\begin{aligned}
\varphi_n(C_r^n C_s^n) &= \varphi_n\left(\sum_{t=0}^{\min(r,s)} \binom{n-t}{r-t} \binom{n-t}{s-t} \binom{2n+k-r-s}{2n+k-n-t} C_t^n \right) \\
&= \sum_{t=0}^{\min(r,s)} \binom{(n-1)-(t-1)}{(r-1)-(t-t)} \binom{(n-1)-(t-1)}{(s-1)-(t-1)} \binom{2(n-1)+k-(r-1)-(s-1)}{2(n-1)+k-(n-1)-(t-1)} \varphi_n(C_t^n) \\
&= \sum_{t=0}^{\min(r,s)} \binom{(n-1)-(t-1)}{(r-1)-(t-t)} \binom{(n-1)-(t-1)}{(s-1)-(t-1)} \binom{2(n-1)+k-(r-1)-(s-1)}{2(n-1)+k-(n-1)-(t-1)} C_{t-1}^{n-1} \\
&= C_{r-1}^{n-1} C_{s-1}^{n-1} = \varphi_n(C_r^n) \varphi_n(C_s^n).
\end{aligned}$$

So φ_n is an algebra epimorphism. Since $\mathrm{Ker}\varphi_n = (C_0^n) = (\oplus_{i=0}^n D_i^n) = (J^*)$,

$$FJ(2(n-1)+k, n-1) \cong FJ(2n+k, n)/(J^*). \quad \square$$

A Johnson scheme $J(m, n)$ is always described with the two parameters m and n. When we consider their modular structures, we do not have to consider the case where m is very large compared with n, by the next lemma.

Lemma 3. *If $p^k > n$, then*

$$FJ(m+p^k, n) \cong FJ(m, n).$$

The proof is omitted, because it is similar to the proof of Theorem 2. From the next lemma, we can determine whether the dimension of the principal block of $FJ(m, n)$ is 1, and if not, we can calculate its dimension. We discuss the method in Section 5.

Lemma 4 ([7]). *The dimension of the principal block b_0^* of $FJ(2n+k, n)$ is 1 if and only if $p \nmid \binom{2n+k}{n}$.*

4. Decompositions of modular adjacency algebras

If the prime $p = 2$, then $FJ(2(2^t-1), 2^t-1)$ can be expressed as a tensor product of some $FJ(2(2-1), 2-1)$.

Theorem 5 ([11]). *Let F be a field whose characteristic is 2. Then*

$$FJ(2(2^t-1), 2^t-1) \cong \bigotimes^t FJ(2(2-1), 2-1) \cong \bigotimes^t F[x]/(x^2).$$

We assume from now on that p is an odd prime as long as there is no claim to the contrary. Let B_i^{2n+l} be the ith intersection matrix of $J(2n+l, n)$:

$$\{B_i^{2n+l}\}_{j,k} := p_{i,j}^k = \sum_{r=0}^{n-k} \binom{n-k}{r}\binom{k}{n-i-r}\binom{k}{n-j-r}\binom{n+l-k}{i+j+r-n},$$

$(0 \le i, j, k \le n)$.

For any $i, j, k \in \{0, 1, 2, \ldots, n\}$, since the equations

$$B_i^{2n+l} \cdot B_j^{2n+l} = \sum_{k=0}^{n} p_{i,j}^k B_k^{2n+l}$$

hold,

$$\oplus_{i=0}^n FB_i^{2n+l}$$

becomes an F-algebra. Let us denote this algebra by FB^{2n+l}. We call this an intersection algebra over F. Since the map between adjacency matrices and intersection matrices $D_i \mapsto B_i$ is an algebra isomorphism,

$$FJ(2n+l, n) \cong FB^{2n+l}.$$

We prove the next theorem by direct modulo calculation [8].

Theorem 6. *For $0 \le c, z \le p-1$, $0 \le d, y \le p^{t-1}-1$,*

(i) *If $y \ge d$ or $z = 0$,*

$$B_{zp^{t-1}+y}^{2(p^t-1)+cp^{t-1}+d} \equiv B_z^{2(p-1)+c} \otimes B_y^{2(p^{t-1}-1)+d}.$$

(ii) *If $y < d$ and $z \ne 0$,*

$$B_{zp^{t-1}+y}^{2(p^t-1)+cp^{t-1}+d} \equiv B_z^{2(p-1)+c+1} \otimes B_y^{2(p^{t-1}-1)+d} + (B_z^{2(p-1)+c} - B_z^{2(p-1)+c+1}) \otimes \sum_{l=y}^{p^{t-1}-1} \gamma_l B_l^{2(p^{t-1}-1)+d},$$

where $\gamma_l (y \le l \le p^{t-1}-1)$ are positive integers.

Using Theorem 6, we prove the next theorem.

Theorem 7. *$FJ(2(p^t-1)+cp^{t-1}+d, p^t-1)$ $(0 \le c \le p-1, 0 \le d \le p^{t-1}-1)$ is a subalgebra of the sum of $FJ(2(p-1)+c, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$ and $FJ(2(p-1)+c+1, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$.*

5. Structure of $FJ(2(p^t-1)+l,p^t-1)$

By Theorem 7 and induction on t, $FJ(2(p^t-1)+l,p^t-1)(0\le l\le p^t-1)$ is a subalgebra of sums of tensor products of $FJ(2(p-1)+h,p-1)(0\le h\le p-1)$. We consider structures of $FJ(2(p-1)+h,p-1)$. In this section, let p be an odd prime.

Lemma 8 ([7]). *Irreducible characters χ_i and χ_j of $K\mathfrak{X}$ belong to the same block if and only if $\chi_i(D_r)\equiv\chi_j(D_r)(\bmod\ (\pi))$ for all $r=0,\dots,d$.*

Lemma 9 ([7]). *The dimension of B_i^* is equal to the number of χ_j belonging to B_i.*

Character values of Johnson schemes are rational integers. Irreducible characters of a Johnson scheme are determined simply by reducing the character table of the Johnson scheme modulo p. Characters χ_i and χ_j are in the same block if and only if all reduced entries in the row of the character table corresponding to χ_i are the same as the reduced entries in the row corresponding to χ_j.

Lemma 10 ([10]). *Let $P^{2(p-1)+h,p-1}$ be the character table for $J(2(p-1)+h,p-1)(0\le h\le p-1)$. A reduced ath row of $P^{2(p-1)+h,p-1}$ is equal to a reduced bth row of $P^{2(p-1)+h,p-1}$ if and only if $a+b\equiv h-1(\bmod\ p)$.*

By this lemma, for an odd prime p, we let p equal $2m+1$ (where m is a positive integer). There are m pairs of equivalence rows and there is one isolated row. This means that $FJ(2(p-1)+h,p-1)$ can be decomposed as m blocks of dimension 2 and one block of dimension 1. We describe $FJ(2(p-1)+h,p-1)$ via the following theorem.

Theorem 11. *For an odd prime p (thus $p=2m+1$), we have*

$$FJ(2(p-1)+h,p-1)\cong m(F[x]/(x^2))\oplus F.$$

Corollary 12 ([6]). *For $0<t\le p/2$,*

$$FJ(p,t)\cong\underbrace{F[x]/(x^2)\oplus F\oplus\cdots\oplus F}_{t}.$$

In particular, $FJ(p,t)$ is a finite-representation type algebra [4].

Each entry of $P^{2(p^t-1)+l,p^t-1}$ is described as some multiplication of entries of character tables of Johnson schemes with smaller parameters [10]. This means that each entry consists of a Cartesian product of entries of t row vectors. We can express an ath row and a bth row of $P^{2(p^t-1)+l,p^t-1}$ as the following:

$$P^{2(p^t-1)+l,p^t-1}(a)\equiv P^{2(p-1)+f_0,p-1}(a_0)\cdots P^{2(p-1)+f_{t-1},p-1}(a_{t-1}),$$
$$P^{2(p^t-1)+l,p^t-1}(b)\equiv P^{2(p-1)+g_0,p-1}(b_0)\cdots P^{2(p-1)+g_{t-1},p-1}(b_{t-1})$$

where $l=l_0+l_1p+\cdots+l_{t-1}p^{t-1}$, $a=a_0+a_1p+\cdots+a_{t-1}p^{t-1}$, $b=b_0+b_1p+\cdots+b_{t-1}p^{t-1}$, $f_0=g_0=l_0$.

$$f_k,g_k=\begin{cases}l_k+1(\bmod\ p) & (a_0+\cdots+a_{k-1}p^{k-1},\ b_0+\cdots+b_{k-1}p^{k-1}<l_0+\cdots+l_{k-1}p^{k-1}),\\ l_k & (a_0+\cdots+a_{k-1}p^{k-1},\ b_0+\cdots+b_{k-1}p^{k-1}\ge l_0+\cdots+l_{k-1}p^{k-1}),\end{cases}$$

$(1\le k\le t-1)$.

Proposition 13 ([10]).

$$P^{2(p^t-1)+l,p^t-1}(a)\equiv P^{2(p^t-1)+l,p^t-1}(b)(\bmod\ p)$$

if and only if

$$P^{2(p-1)+f_k}(a_k)\equiv P^{2(p-1)+g_k}(b_k)(\bmod\ p),\quad(0\le k\le t-1).$$

Theorem 14 ([10]). *For any $l(\ge0)$, the number of blocks of $FJ(2(p^t-1)+l,p^t-1)$ and their dimensions are constant. They depend only on t.*

Proof. We let $p=2m+1$. Let the p-adic expansion of a be $a_0+a_1p+\cdots+a_{t-1}p^{t-1}$, and the ath row be expressed as

$$P^{2(p^t-1)+l,p^t-1}(a)\equiv P^{2(p-1)+f_0,p-1}(a_0)\cdots P^{2(p-1)+f_{t-1},p-1}(a_{t-1})$$

($0 \le f_k \le p-1, 0 \le k \le t-1$). We suppose that u is the number of factors of the right-hand side, which do not have equivalence rows in each character table. By Lemma 10, the a_kth row of $P^{2(p-1)+f_k,p-1}$ does not have equivalence rows in $P^{2(p-1)+f_k,p-1}$ if and only if there is no integer $b (0 \le b \le p-1)$ such that

$$a_k + b \equiv f_k - 1 (\bmod p).$$

For the other $t-u$ factors of the right-hand side, there is only one such integer. By Proposition 13, the number of ways to choose such a is m^{t-u} since there are m equivalence rows for each of the $t-u$ factors, for a fixed u. Then, there are 2^{t-u} rows which are equivalent to the ath row, including the ath row. Since the number of such u is $\binom{t}{u}$, there are $\binom{t}{u} m^{t-u}$ sets, which have 2^{t-u} equivalence rows in $P^{2(p^t-1)+l,p^t-1}$. In particular, the following equation holds:

$$\sum_{u=0}^{t} 2^{t-u} \binom{t}{u} m^{t-u} = (2m+1)^t = p^t.$$

So the number of blocks and their dimensions are independent of l. $\square$

In Theorem 6, if $d = 0$, then $FJ(2(p^t-1) + cp^{t-1}, p^t-1)$ can be expressed as tensor products of $FJ(2(p-1)+c, p-1)$ and some $FJ(2(p-1), p-1)$.

Corollary 15. *For $0 \le c \le p-1$,*

$$\begin{aligned} FJ(2(p^t-1) + cp^{t-1}, p^t-1) &\cong FJ(2(p-1)+c, p-1) \bigotimes^{t-1} FJ(2(p-1), p-1) \\ &\cong \bigoplus_{i=0}^{t-1} \left\{ \binom{t}{i} m^{t-i} \bigotimes^{t-i} F[x]/(x^2) \right\} \bigoplus F. \end{aligned}$$

If $d \ne 0$, the next theorem holds by Theorem 7 and the preceding theorem.

Theorem 16. *For $FJ(2(p^t-1) + cp^{t-1} + d, p^t-1)$, let block decompositions of the following two algebras be*

$$FJ(2(p-1)+c, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1) \cong V_0 \oplus \cdots \oplus V_s,$$
$$FJ(2(p-1)+c+1, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1) \cong W_0 \oplus \cdots \oplus W_s.$$

Then, the block decomposition of $FJ(2(p^t-1) + cp^{t-1} + d, p^t-1)$ consists of some of $V_0, \ldots, V_s, W_0, \ldots, W_s$.

Proof. Let X be $FJ(2(p^t-1) + cp^{t-1} + d, p^t-1)$, A be $FJ(2(p-1)+c, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$ and B be $FJ(2(p-1)+c+1, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$. The block decomposition of X may be expressed as $U_0 \oplus \ldots \oplus U_s$, where $U_i = e_i X$ and $\{e_i\}_{0 \le i \le s}$ are primitive idempotents of X. Let α be $X \cap A$ and β be $X \cap B$. By Theorem 6, α, β are not empty. α is an (α, α)-module; β is a (β, β)-module. Let $\hat{\alpha}$ be a minimum F-dimensional (X, X)-submodule which contains α. $\hat{\alpha}$ is an (α, α)-module. Similarly, we define a (X, X)-submodule $\hat{\beta}$. Then, $\hat{\alpha} \cap U_i$ is a (α, α)-module and an (X, X)-submodule. Similarly, $\hat{\beta} \cap U_i$ is a (β, β)-module and an (X, X)-submodule. Since $U_i = (\hat{\alpha} \cap U_i) + (\hat{\beta} \cap U_i)$ and $\alpha \cap \beta = 1 \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$,

$$(\hat{\alpha} \cap U_i) \cap (\hat{\beta} \cap U_i) = (\hat{\alpha} \cap \hat{\beta}) \cap U_i = (\alpha \cap \beta) \cap U_i = 0.$$

So $U_i = (\hat{\alpha} \cap U_i) \oplus (\hat{\beta} \cap U_i)$. Since U_i is an indecomposable (X, X)-module, $\hat{\alpha} \cap U_i = U_i$ and $\hat{\beta} \cap U_i = 0$ or $\hat{\beta} \cap U_i = U_i$ and $\hat{\alpha} \cap U_i = 0$. If $\hat{\alpha} \cap U_i = U_i$ and $\hat{\beta} \cap U_i = 0$,

$$U_i = \hat{\alpha} \cap U_i = e_i X = e_i \hat{\alpha} = e_i \alpha \subset e_i A = V_{i_1} \oplus \cdots \oplus V_{i_r}.$$

By Theorem 14, for any c and d, the number of blocks and their F-dimension are constants and they are the same as for A and B. This means that

$$e_i \alpha = e_i X = e_i A = U_i = V_i. \quad \square$$

By induction on t, our assumption is

$$FJ(2(p^{t-1}-1)+d, p^{t-1}-1) \cong \bigoplus_{i=0}^{t-2} \left\{ \binom{t}{i} m^{t-i} \bigotimes^{t-i} F[x]/(x^2) \right\} \bigoplus F.$$

Then, independently of c and d, by Theorem 16, $FJ(2(p^t-1) + cp^{t-1} + d, p^t-1)$ consists of some blocks of $FJ(2(p-1) + c, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$ and $FJ(2(p-1)+c+1, p-1) \otimes FJ(2(p^{t-1}-1)+d, p^{t-1}-1)$. Since the number of blocks and their dimensions are the same,

$$FJ(2(p^t-1) + cp^{t-1} + d, p^t-1) \cong \bigoplus_{i=0}^{t-1} \left\{ \binom{t}{i} m^{t-i} \bigotimes^{t-i} F[x]/(x^2) \right\} \bigoplus F.$$

By Lemma 4, for $FJ(2n+k, n)$, if $p \nmid \binom{2n+k}{n}$, then $\mathrm{Ker}\varphi_n$ is the principal block and it is isomorphic to F. Let $FJ(2n+k, n)$ be decomposed as $b_0^* \oplus b_1^* \oplus \cdots \oplus b_s^*$. Then

$$FJ(2(n-1)+k, n-1) \cong FJ(2n+k, n)/\mathrm{Ker}\varphi_n \cong b_1^* \oplus \cdots \oplus b_s^*.$$

On the other hand if $p \mid \binom{2n+k}{n}$, then $\mathrm{Ker}\varphi_n = (J^*)$ is an ideal of a radical of b_0^* whose dimension is 1.

$$FJ(2(n-1)+k, n-1) \cong FJ(2n+k, n)/\mathrm{Ker}\varphi_n \cong B_0/(J^*) \oplus B_1 \oplus \cdots \oplus B_s.$$

By results of [10], we can calculate the dimensions of principal blocks even if they are not 1.

Lemma 17. *The dimension of the principal block of $FJ(2n+l, n)$ is the number of $(x+p^t-1-n)$th rows of the reduced character table of $J(2(p^t-1)+l, p^t-1)$ which are equivalent to the (p^t-1-n)th row ($0 \le x \le n$).*

Example 5.1. We consider the structure of $FJ(13, 6)$ where $\mathrm{char}F = 3$.

(i) Consider the structure of $FJ(2(p^t-1)+cp^{t-1}+d, p^t-1)$:

$$FJ(2(3^2-1)+0\cdot 3+1, 3^2-1) \cong F[x_1, x_2]/(x_1^2, x_2^2) \oplus F[x_3]/(x_3^2) \oplus F[x_4]/(x_4^2) \oplus F.$$

(ii) Consider the structures of $FJ(2(p^t-1-k)+cp^{t-1}+d, p^t-1-k)$, $(k = 1, 2, \ldots)$ by induction:

Since $3 \nmid \binom{17}{8}$, the dimension of the principal block of $FJ(17, 8)$ is 1.

$$FJ(2(3^2-1-1)+1, 3^2-1-1) \cong F[x_1, x_2]/(x_1^2, x_2^2) \oplus F[x_3]/(x_3^2) \oplus F[x_4]/(x_4^2).$$

Since $3 \mid \binom{15}{7}$, the dimension of the principal block of $FJ(15, 7)$ is not 1. By the reduced character table of $J(17, 8)$, $\dim_F b_0^* = 4$. So $F[x_1, x_2]/(x_1^2, x_2^2)$ is the principal block of $FJ(15, 7)$.

$$FJ(2(3^2-1-2)+1, 3^2-1-2) \cong F[x_1, x_2]/(x_1^2, x_1x_2, x_2^2) \oplus F[x_3]/(x_3^2) \oplus F[x_4]/(x_4^2).$$

Remark 1. If p is the even prime 2, then Theorem 14 does not hold because $FJ(2(p-1), p-1)$ is local and $FJ(2(p-1)+1, p-1)$ is semisimple. We need to consider the case of $p = 2$ using other methods.

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